JOURNAL OF MATERIALS SCIENCE **39** (2004) 745–746

On the fundamental relations used in the analysis of nanoindentation data

GUANGHUI FU
Lam Research Corporation, 4540 Cushing Parkway, Fremont, CA 94538, USA
E-mail: guanghui.fu@lamrc.com

Nanoindentation experiments have become a commonly used technique to investigate mechanical properties of thin films and small volumes of materials. The analysis of the experimental load—displacement (P-h) curve is based on the fundamental relationship among contact stiffness, contact area and elastic modulus. The slope of the P-h curve, $S = \mathrm{d}P/\mathrm{d}h$, is defined as contact stiffness and it can be measured from nanoindentation experiments. The fundamental relationship relates contact stiffness to the projected contact area (A), Young's modulus of the material (E), and Poisson's ratio of the material (ν), as

$$S = \frac{2}{\sqrt{\pi}} \frac{E}{(1-\nu^2)} \sqrt{A} \tag{1}$$

The fundamental relationship, which is used in the interpretation of nanoindentation experimental data, is based on the analytical solution of normal indentation of an elastic half-space by a smooth frictionless axisymmetric indenter. This relationship has been verified for an indenter whose shape is flat-ended, conical, or parabolic. Pharr, Oliver and Brotzen show that this relationship holds true if the indenter profile can be described as a solid of revolution of a smooth function [1]. However, their proof is not mathematically correct (see Appendix). This relationship is revisited in this paper. The indenter profile, a smooth function, is expanded as a Maclaurin series and the derivation shows that the fundamental relationship is valid.

We consider a rigid smooth frictionless axisymmetric indenter with its axis of revolution as the z-axis indenting normally into the plane $z = 0$ of an elastic half-space $z \geq 0$. The problem is considered in the linear theory of elasticity and the half-space is assumed to be isotropic and homogeneous. The contact region between the indenter and the half-space is simply connected.

The following equations give the relevant displacement and stresses for the half-space. The vertical component of the displacement is denoted by u_z, and the stress components have two subscripts corresponding to the appropriate coordinates. E and ν are Young's modulus and Poisson's ratio for the half-space.

As Fig. 1 shows, the boundary conditions for the half-space at $z = 0$ are

$$\tau_{zr} = \tau_{z\theta} = 0, \quad (0 \leq r < \infty) \tag{2}$$

$$\sigma_{zz} = 0, \quad (r > a) \tag{3}$$

$$u_z = h + \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha r^\alpha, \quad (0 \leq r \leq a) \tag{4}$$

where α is a positive real number. The second term at the right hand side of Equation 4 describes the indenter shape.

The radius of the contact area, a, and the indentation depth, h, are related by the following equation [2]:

$$\frac{2}{\sqrt{\pi}} h + \sum_{\alpha=\alpha_1}^{\alpha_n} (1+\alpha) \cdot a_\alpha \cdot \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)} a^\alpha = 0 \tag{5}$$

The total vertical load, P, which causes the displacement h is

$$P = \sqrt{\pi} \frac{E}{1-\nu^2} \left[\frac{2}{\sqrt{\pi}} ah + \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha \cdot \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)} a^{1+\alpha} \right] \tag{6}$$

The indenter profile, $f(r)$, is a smooth function, and can be expanded as a Maclaurin series:

$$f(r) = \sum_{i=1}^{\infty} \frac{f^{(\mathrm{i})}(0)}{i!} r^{\mathrm{i}} \tag{7}$$

The corresponding displacement equation for the indenter is

$$u_z = h + f(r) = h + \sum_{i=1}^{\infty} \frac{f^{(\mathrm{i})}(0)}{i!} r^{\mathrm{i}}, \quad (0 \leq r \leq a) \tag{8}$$

Note Equation 5 is true for any positive real number α. From Equation 5, we have

$$\frac{2}{\sqrt{\pi}} h + \sum_{i=1}^{\infty} (1+i) \cdot \frac{f^{(\mathrm{i})}(0)}{i!} \cdot \frac{\Gamma((2+i)/2)}{\Gamma((3+i)/2)} a^{\mathrm{i}} = 0 \tag{9}$$

The corresponding load-displacement relation is

$$P = \sqrt{\pi} \frac{E}{1-\nu^2} \left[\frac{2}{\sqrt{\pi}} ah + \sum_{i=1}^{\infty} \cdot \frac{\Gamma((2+i)/2)}{\Gamma((3+i)/2)} \frac{f^{(\mathrm{i})}(0)}{i!} a^{1+\mathrm{i}} \right] \tag{10}$$

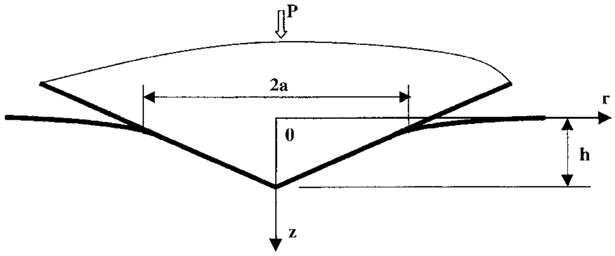

Figure 1 Normal indentation of an elastic half-space.

And the contact stiffness is

$$S = \frac{\mathrm{d}p}{\mathrm{d}h} = \sqrt{\pi}\frac{E}{1-\nu^2}\left\{\frac{2}{\sqrt{\pi}}\left(a + h\frac{\mathrm{d}a}{\mathrm{d}h}\right) + \sum_{i=1}^{\infty}\left[(1+i)\frac{\Gamma((2+i)/2)}{\Gamma((3+i)/2)}\frac{f^{(i)}(0)}{i!}a^{i}\right]\frac{\mathrm{d}a}{\mathrm{d}h}\right\} \quad (11)$$

Noting Equations 9 and 11 becomes

$$S = 2\frac{E}{1-\nu^2}a \quad (12)$$

Thus,

$$S = \frac{2}{\sqrt{\pi}}\frac{E}{1-\nu^2}\sqrt{A} \quad (13)$$

Appendix

There are several ways to show Pharr *et al.*'s proof [1] is not correct:

a. Equation A2 implies $f'(x)\,\mathrm{d}x = f'(\rho)\,\mathrm{d}\rho$, which is not true unless $x = \rho$.

b. The variable x of the function $f(x)$ is defined at the interval [0, 1] at Equation A1. It becomes [0, a] at Equations A2 and A3. They are not the same unless $a = 1$.

c. The correct derivation for Equation A2 should be

$$\frac{4\mu a}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}a}\int_{x=0}^{x=1}\sqrt{1-x^2}f'(x)\,\mathrm{d}x = \frac{4\mu a}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}a}\int_{\rho=0}^{\rho=a}\sqrt{1-\left(\frac{\rho}{a}\right)^2}f'\left(\frac{\rho}{a}\right)d\frac{\rho}{a} = \frac{4\mu a}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}a}\int_{\rho=0}^{\rho=a}\frac{\sqrt{1-(\frac{\rho}{a})^2}}{a}f'\left(\frac{\rho}{a}\right)\mathrm{d}\rho$$

The correct derivation for Equation A3 should be

$$\frac{4\mu a}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}a}\int_{x=0}^{x=1}\sqrt{1-x^2}f'(x)\,\mathrm{d}x = \frac{4\mu a}{(1-\nu)}\int_{\rho=0}^{\rho=a}\frac{\partial}{\partial a}\left[\frac{\sqrt{1-(\frac{\rho}{a})^2}}{a}f'\left(\frac{\rho}{a}\right)\right]\mathrm{d}\rho = \frac{4\mu a}{(1-\nu)}\int_{\rho=0}^{\rho=a}\left\{\frac{\partial}{\partial a}\left[\frac{\sqrt{1-(\frac{\rho}{a})^2}}{a}\right]f'\left(\frac{\rho}{a}\right) + \frac{\sqrt{1-(\frac{\rho}{a})^2}}{a}\frac{\partial}{\partial a}\left[f'\left(\frac{\rho}{a}\right)\right]\right\}\mathrm{d}\rho$$

It is obvious that it will not lead to Equation A4. Thus, the authors failed to prove the fundamental relation.

References

1. G. M. PHARR, W. C. OLIVER and F. R. BROTZEN, *J. Mater. Res.* **7** (1992) 613.
2. G. FU and A. CHANDRA, *J. Appl. Mech. ASME* **69** (2002) 142.

Received 28 August and accepted 8 September 2003